Involutory Functions and Moore-Penrose Inverses of Matrices in an Arbitrary Field

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ABSTRACT

Let F be a field, and M be the set of all matrices over F. A function f from M into M, which we write $f(A)=A^s$ for $A\in M$, is involutory if (1) $(AB)^s=B^sA^s$ for all A,B in M whenever the product AB is defined, and (2) $(A^s)^s=A$ for all $A\in M$. If f is an involutory function on M, then A^s is $n\times m$ if A is $m\times n$; furthermore, Rank $A=$ Rank A^s, the restriction of f to F is an involutory automorphism of F, and $(aA+bB)^s=a^sA^s+b^sB^s$ for all $m\times n$ matrices A and B and all scalars a and b. For an $A\in M$, an $\tilde{A}\in M$ is called a Moore-Penrose inverse of A relative to f if (i) $A\tilde{A}A=A$, $\tilde{A}A\tilde{A}=\tilde{A}$ and (ii) $(A\tilde{A})^s=A\tilde{A}$, $(\tilde{A}A)^s=\tilde{A}A$. A necessary and sufficient condition for A to have a Moore-Penrose inverse relative to f is that Rank $A=$ Rank $AA^s=$ Rank A^sA. Furthermore, if an involutory function f preserves circulant matrices, then the Moore-Penrose inverse of any circulant matrix relative to f is also circulant, if it exists.

1. INTRODUCTION

Recall that the Moore-Penrose inverse of an $m\times n$ complex matrix A is the $n\times m$ matrix $\tilde{A}$ such that (i) $A\tilde{A}A=A$, $\tilde{A}A\tilde{A}=\tilde{A}$ and (ii) $(\tilde{A}A)^*=\tilde{A}A$, $(A\tilde{A})^*=A\tilde{A}$, where the asterisk denotes conjugate transpose. There are many ways to extend the idea of the Moore-Penrose inverse of a matrix from the complex number field C to an arbitrary field F [1, 2, 5]. For example, we can keep condition (i) and replace the conjugation of the complex numbers by an involutory automorphism of F [4]. In the case of complex numbers, when A is

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considered as a linear transformation from C^n to C^m, then A^* is the adjoint of A relative to the standard inner product of C^n and C^m. Thus $\tilde{A}$ is the Moore-Penrose inverse of A if $\tilde{A}A$ and $A\tilde{A}$ are self-adjoint in C^n and C^m respectively. From this point of view, if A is an $m \times n$ matrix over a field F and the vector spaces F^n and F^m are equipped with nondegenerate symmetric (or skew-symmetric) bilinear forms B_n and B_m respectively, then A induces an $n \times m$ matrix A^s, the adjoint of A, where $B_m(Ax, y) = B_n(x, A^s y)$ for all $x \in F^n$ and all $y \in F^m$. In this case we call an $n \times m$ matrix $\tilde{A}$ a Moore-Penrose inverse of A relative to B_n and B_m if $(\tilde{A}A)^s = \tilde{A}A$ and $(A\tilde{A})^s = A\tilde{A}$. In other words, $\tilde{A}A$ and $A\tilde{A}$ are self-adjoint in F^n and F^m respectively. But both cases can be obtained from an involutory function defined on the set of all matrices over F. It is the purpose of this paper to investigate this type of function and the Moore-Penrose inverse of a matrix relative to it. We also show that if an involutory function preserves circulant matrices, then the Moore-Penrose inverse relative to it (if it exists) of any circulant matrix is also circulant. Finally, we give a method which is efficient by using a computer to calculate the Moore-Penrose inverse of a matrix relative to an involutory function.

2. INVOLUTORY FUNCTIONS

Let F be a field and M be the set of all matrices over F. A function f from M into M is involutory, and we write $f(A) = A^s$ for $A \in M$, if (1) $(AB)^s = B^s A^s$ for all A, B in M whenever the product AB is defined, and (2) $(A^s)^s = A$ for all A in M.

Example. Let $a \to \bar{a}$ be an involutory automorphism of the field F. For any matrix A, we let $A^* = (\bar{a}_{ij})^t$. An $n \times n$ matrix B_n is hermitian (skew-hermitian) if $B_n^* = B_n$ $(-B_n)$. For each n, let B_n be an $n \times n$ invertible hermitian (skew-hermitian) matrix. For any $m \times n$ matrix A, we define

$$f(A) = A^s = B_n^{-1} A^* B_m.$$

Then f is an involutory function. Notice that if all B_n are the identity matrices, then $A^s = A^*$. Also, if $a = \bar{a}$ for all $a \in F$, then $A^s = B_n^{-1} A^t B_m$ is the adjoint of A relative to the nondegenerate symmetric (skew-symmetric) bilinear forms induced by B_n and B_m in F^n and F^m respectively which we mentioned in the introduction.

THEOREM 1. *Let f be an involutory function defined on the set of all matrices over a field F. Then*

(1) *$f(A)=A^s$ is an $n\times m$ matrix if A is an $m\times n$ matrix.*
(2) Rank $A^s=$ Rank A.
(3) *The restriction of f to F is an involutory automorphism of F.*
(4) *If A and B are $m\times n$ matrices and a and b are elements of F, then $(aA+bB)^s=a^sA^s+b^sB^s$.*

Proof. We first show that all identity matrices are fixed under f. Then it follows that $A=I_mAI_n$ and $A^s=I_nA^sI_m$; hence A^s must be an $n\times m$ matrix. Since $I_n=I_nI_n$, therefore $I_n^s=I_n^sI_n^s$ and I_n^s must be a square ma. Let I_n^s be $h\times h$. For any $h\times h$ ma B_h, $(I_n^sB_h)^s=B_h^s$. This implies $I_n^s=I_h$, the $h\times h$ identity matrix. If C is an $n\times h$ matrix, then $C=I_nCI_h$ and $C^s=I_h^sC^sI_n^s=I_nC^sI_h$. Hence C^s must be also $n\times h$ (and similarly for $h\times n$ matrices). Suppose that $n\geqslant h$. Let H be the $h\times n$ matrix $(I_h|0)$. Then $HH^t=I_h$ and $(H^t)^sH^s=I_h^s=I_n$. The rank of $(H^t)^s$ and the rank of H^s are at most h, and the rank of I_n is n. This implies $n\leqslant h$. Thus $n=h$ and $I_n^s=I_n$.

Let A be an $m\times n$ matrix of rank r, and P and Q be $m\times m$ and $n\times n$ invertible matrices such that

$$PAQ=\left(\begin{array}{c|c} I_r & 0\\ \hline 0 & 0\end{array}\right).$$

Then

$$Q^sA^sP^s=\left(\begin{array}{c|c} I_r & 0\\ \hline 0 & 0\end{array}\right)^s.$$

Since the identity matrices are fixed under f, P^s and Q^s are also invertible. Therefore

$$\text{Rank } A^s=\text{Rank } Q^sA^sP^s=\text{Rank}\left(\begin{array}{c|c} I_r & 0\\ \hline 0 & 0\end{array}\right)^s.$$

Let H be the $r\times m$ matrix $(I_r|0)$ and K be the $n\times r$ matrix $\left(\begin{array}{c} I_r\\ \hline 0\end{array}\right)$. Then

$$H\left(\begin{array}{c|c} I_r & 0\\ \hline 0 & 0\end{array}\right)K=I_r.$$

Thus

$$K^s\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)^s H^s = I_r^s = I_r.$$

This implies

$$\operatorname{Rank} A^s = \operatorname{Rank}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)^s \geqslant \operatorname{Rank} I_r = r = \operatorname{Rank} A.$$

Therefore $\operatorname{Rank} A = \operatorname{Rank}(A^s)^s \geqslant \operatorname{Rank} A^s \geqslant \operatorname{Rank} A$ and hence $\operatorname{Rank} A = \operatorname{Rank} A^s$. Consequently, f sends zero matrices to zero matrices.

Let $e_1, \dots, e_n$ be the set of $1 \times n$ matrices which are the elements of the standard basis of F^n when they are considered as row vectors, and also let β_i be the transpose of e_i. Since $e_i\beta_j = \delta_{ij}$ and $(e_i\beta_j)^s = \beta_j^s e_i^s = \delta_{ij}$, the set $\{e_i^s\}$, considered as column vectors in F^n, is also a basis of F^n (the matrix E with columns e_i^s is invertible with inverse E^{-1} the matrix with row vectors β_j^s).

Let E_{ij} be the matrix units, that is, E_{ij} is the matrix with 1 at (i, j) position and zero elsewhere. For any matrix $A = (a_{ij})$, $A = \sum_{i,j} a_{ij}E_{ij}$. Hence $I_n = E_{11} + \cdots + E_{nn}$. Since $e_iE_{jj} = (0, \dots, 0)$ if $i \neq j$ and $e_iE_{jj} = e_i$ if $i = j$, we have

$$E_{jj}^s e_i^s = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{if } i \neq j \quad \text{and} \quad E_{jj}^s e_i^s = e_i^s, \quad \text{if } i = j.$$

This implies $(\sum_{j=1}^n E_{jj}^s)e_i^s = e_i^s$ for all $i = 1, \dots, n$. Hence $\sum_{j=1}^n E_{jj}^s = I_n = I_n^s$.

For all x, y in F,

$$(x \quad y) = (x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned}
(x \quad y)^s &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^s (x \quad y)^s \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^s (x \quad y)^s + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^s (x \quad y)^s \\
&= \left[(x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]^s + \left[(x \quad y)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]^s \\
&= (x \quad 0)^s + (0 \quad y)^s.
\end{aligned}$$

Since

$$(x+y)^s = \left[\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^s \begin{pmatrix} x & y \end{pmatrix}^s = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^s \begin{pmatrix} x & 0 \end{pmatrix}^s + \begin{pmatrix} 1 \\ 1 \end{pmatrix}^s \begin{pmatrix} 0 & y \end{pmatrix}^s$$

$$= \left[\begin{pmatrix} x & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^s + \left[\begin{pmatrix} 0 & y \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^s$$

$$= x^s + y^s,$$

the restriction of f to F is an involutory automorphism of F.

Let $C = cI_n$ be an $n \times n$ scalar matrix. We are going to show that $C^s = c^s I_n$. Since $CA = AC$ for all $n \times n$ matrices A and $A^s C^s = C^s A^s$, C^s is also a scalar matrix. Let $C^s = dI_n$,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^s = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^s = \begin{pmatrix} k_1 & \cdots & k_n \end{pmatrix}.$$

Then

$$\left[\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]^s = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^s$$

$$= h_1 k_1 + \cdots + h_n k_n = 1^s = 1$$

and

$$\left[\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}(cI_n)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]^s = c^s = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^s (cI_n)^s \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^s$$

$$= d(h_1 k_1 + \cdots + h_n k_n) = d.$$

Hence $(cI_n)^s = c^s I_n$. For any $m \times n$ matrix A and scalar a, $(aA)^s = (aI_m A)^s = A^s a^s I_m = a^s A^s$.

Let A be an $m \times n$ matrix. Then $A = I_m A I_n$ and $A^s = (\sum_{i,j}^{m,n} a_{ij} E_{ij})^s = I_n^s A^s I_m^s = (\sum_{j=1}^{n} E_{jj}^s) A^s (\sum_{i=1}^{m} E_{ii}^s) = \sum_{i,j}^{m,n} (E_{ii} A E_{jj})^s = \sum_{i,j}^{m,n} (a_{ij} E_{ij})^s = \sum_{i,j}^{m,n} (a_{ij})^s E_{ij}^s$. If $A = \sum_{i,j}^{m,n} a_{ij} E_{ij}$ and $B = \sum_{i,j}^{m,n} b_{ij} E_{ij}$ are $m \times n$ matrices and a and b are scalars, then $(aA + bB)^s = [\sum_{i,j}^{m,n} (aa_{ij} + bb_{ij}) E_{ij}]^s = \sum_{i,j}^{m,n} (aa_{ij} + bb_{ij})^s E_{ij}^s = a^s \sum_{i,j}^{m,n} (a_{ij})^s E_{ij}^s + b^s \sum_{i,j}^{m,n} (b_{ij})^s E_{ij}^s = a^s A^s + b^s B^s$. This completes the proof of Theorem 1. ∎

If we already have an involutory function f defined on M, then we can construct another involutory function g in the following way. For each n, let B_n be an $n \times n$ invertible f-hermitian (skew f-hermitian) matrix, that is, $f(B_n) = B_n^s = B_n$ $(-B_n)$. For any $m \times n$ matrix A we define

$$g(A) = B_n^{-1} A^s B_m.$$

Then g is also involutory. As an open question: is every involutory function obtained in this manner?

3. MOORE-PENROSE INVERSES

Let f be in involutory function defined on the set of all matrices over a field F. For a matrix A, a matrix $\tilde{A}$ is called a Moore-Penrose inverse of A relative to f if (i) $A\tilde{A}A = A$, $\tilde{A}A\tilde{A} = \tilde{A}$ and (ii) $(\tilde{A}A)^s = \tilde{A}A$, $(A\tilde{A})^s = A\tilde{A}$.

Theorem 2. *If A has a Moore-Penrose inverse relative to an involutory function f, then it is unique.*

Proof. Let $\tilde{A}$ and B be Moore-Penrose inverses of A relative to f. Then $\tilde{A} = \tilde{A}A\tilde{A} = \tilde{A}(ABA)\tilde{A} = (\tilde{A}A)(BA)\tilde{A} = (\tilde{A}A)^s(BA)^s\tilde{A} = (BA\tilde{A}A)^s\tilde{A} = (BA)^s\tilde{A} = BA\tilde{A}$. On the other hand, $B = BAB = B(A\tilde{A}A)B = B(A\tilde{A})(AB) = B(A\tilde{A})^s(AB)^s = B(ABA\tilde{A})^s = B(A\tilde{A})^s = BA\tilde{A}$. Hence $\tilde{A} = B$. ∎

Theorem 3. *A matrix A has a Moore-Penrose inverse relative to f if and only if* Rank A = Rank AA^s = Rank A^sA.

Proof. Let A be an $m \times n$ matrix over a field F and $V = F^n, W = F^m$. Then A can be considered a linear transformation from V into W, and A^s a linear transformation from W into V. Let N_A, N_{A^s} and R_A, R_{A^s} be the corresponding nullspaces and ranges of A and A^s respectively. The existence of $\tilde{A}$ is equivalent to $V = N_A \oplus R_{A^s}$ and $W = R_A \oplus N_{A^s}$ by a theorem of R. Puystijens and D. W. Robinson [2, Theorem 4, p. 137]. But for completeness we will show the construction of $\tilde{A}$ here also.

If A has a Moore-Penrose inverse $\tilde{A}$ relative to f, then $(A\tilde{A})^s = \tilde{A}^s A^s = A\tilde{A}$. Hence $A = A\tilde{A}A = \tilde{A}^s A^s A$. This implies Rank A = Rank $A^s A$. We can also verify that if $\tilde{A}$ is the Moore-Penrose inverse of A, then $(\tilde{A})^s$ is the Moore-Penrose inverse of A^s. Hence Rank A^s = Rank$(A^s)^s A^s$ = Rank AA^s. By Theorem 1, Rank A = Rank A^s = Rank AA^s.

Suppose that Rank A = Rank AA^s. Since Rank $AA^s = \dim A(R_{A^s}) \leqslant$ Rank A^s = Rank A, therefore if Rank AA^s = Rank A, then A must be injective on R_{A^s}. This means that $N_A \cap R_{A^s} = \{0\}$. From Rank A = Rank $A^s = \dim R_{A^s}$ and $\dim V = \dim N_A +$ Rank A, we have $V = N_A \oplus R_{A^s}$. Similarly, if Rank A = Rank A^s = Rank $A^s A$, then $W = R_{(A^s)^s} \oplus N_{A^s} = R_A \oplus N_{A^s}$. For each $y \in W$ there exists a unique $y_1 \in R_A$ and $y_2 \in N_{A^s}$ such that $y = y_1 + y_2$. Since $R_A = A(R_{A^s})$ and $N_A \cap R_{A^s} = \{0\}$, there exists a unique $x \in R_{A^s}$ where $y_1 = Ax$. The mapping $\tilde{A}$ from W into V defined by $\tilde{A}y = x$ can be verified easily to be linear, and $A\tilde{A}A = A$, $\tilde{A}A\tilde{A} = \tilde{A}$. From $\tilde{A}(A(A^s y)) = A^s y$ for all $y \in W$, we have $A^s = \tilde{A}AA^s$ and hence $A = A(\tilde{A}A)^s$. Thus $\tilde{A}A = \tilde{A}A(\tilde{A}A)^s$ and $(\tilde{A}A)^s = \tilde{A}A$. Also from $A^s y = A^s y_1 = A^s Ax = A^s A\tilde{A}y$ for all $y \in W$, we have $A^s = A^s A\tilde{A}$ and hence $A = (A\tilde{A})^s A$. Thus $A\tilde{A} = (A\tilde{A})^s(A\tilde{A})$ and $(A\tilde{A})^s = A\tilde{A}$. This completes the proof that $\tilde{A}$ is the Moore-Penrose inverse of A relative to f. ∎

Let's return to the example we mentioned before, where for each $m \times n$ matrix A,

$$f(A) = A^s = B_n^{-1} A^* B_m,$$

where $A^* = (\bar{a}_{ij})^t$, and B_n and B_m are $n \times n$ and $m \times m$ invertible hermitian (skew-hermitian) matrices. Then Rank AA^s = Rank $AB_n^{-1}A^*B_m$ = Rank $AB_n^{-1}A^*$ and Rank A^sA = Rank $B_n^{-1}A^*B_mA$ = Rank A^*B_mA. Thus we have the following corollary.

Corollary. *Let F be a field with an involutory automorphisms $a \to \bar{a}$, and B_n and B_m be $n \times n$ and $m \times m$ invertible hermitian (or skew-hermitian) matrices respectively, that is, $B^* = (\bar{b}_{ij})^t = B(\text{or } -B)$. Then for any $m \times n$ matrix A there exists a unique $n \times m$ matrix $\tilde{A}$ such that (i) $A\tilde{A}A = A$, $\tilde{A}A\tilde{A} = \tilde{A}$ and (ii) $A\tilde{A} = B_m^{-1}(A\tilde{A})^*B_m$ and $\tilde{A}A = B_n^{-1}(\tilde{A}A)^*B_n$ if and only if* Rank A = Rank A^*B_mA = Rank $AB_n^{-1}A^*$.

If B_n and B_m are the identity matrices, then this corollary is the classic result of M. H. Pearl [1, Theorem 1, p. 573].

4. CIRCULANT MATRICES

Let A be an $n \times n$ circulant matrix with the first row $(a_0\ a_1\ \cdots\ a_{n-1})$ over a field F. Then $A = a_0 I + a_1 C + \cdots + a_{n-1}C^{n-1}$, where C is the $n \times n$ circulant matrix with the first row $(0\ 1\ 0\ \cdots\ 0)$. Any $n \times n$ matrix is circulant if and only if it commutes with C [3, 7]. It is well known that the Moore-Penrose inverse of a real or complex circulant matrix is also circulant [3]. But this result is no longer true in our situation. For example, let F be the field of integers modulo 2 and $f(A) = A^s = B_3^{-1}A^t B_3$ for all 3×3 matrices A, where

$$B_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The Moore-Penrose inverse of

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

relative to f is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which is not circulant. The reason is that this particular involutory function does not preserve the circulant property. If an involutory function f where $f(C) = C^s$ is circulant, then by Theorem 1, $f(A) = A^s$ is also circulant for any circulant matrix A.

THEOREM 4. *Let f be an involutory function defined on all matrices over a field F such that $f(C) = C^s$ is also circulant, where C is the $n \times n$ circulant matrix with the first row $(0\ 1\ 0\ \cdots\ 0)$. Let A be any $n \times n$ circulant matrix. Then A^s and the Moore-Penrose inverse $\tilde{A}$ of A relative to f, if it exists, are also circulant.*

Proof. Let A be a circulant matrix and $\tilde{A}$ be its Moore-Penrose inverse relative to f. We have pointed out that A^s is also circulant. From

$$\begin{aligned}(A\tilde{A})C^s &= (A\tilde{A})^s C^s = \tilde{A}^s A^s C^s = \tilde{A}^s C^s A^s = \tilde{A}^s C^s (A\tilde{A}A)^s = \tilde{A}^s C^s A^s (A\tilde{A})^s \\ &= \tilde{A}^s A^s C^s A\tilde{A} = A\tilde{A}AC^s\tilde{A} = AC^s\tilde{A} = C^s(A\tilde{A})\end{aligned}$$

and

$$C^s(\tilde{A}A) = C^s(\tilde{A}A)^s = C^sA^s\tilde{A}^s = A^sC^s\tilde{A}^s = (A\tilde{A}A)^sC^s\tilde{A}^s = \tilde{A}AA^sC^s\tilde{A}^s$$
$$= \tilde{A}AC^sA^s\tilde{A}^s = \tilde{A}AC^s\tilde{A}A = \tilde{A}C^sA = (\tilde{A}A)C^s,$$

we have that C^s commutes with $A\tilde{A}$ and $\tilde{A}A$. Hence C commutes with $A\tilde{A}$ and $\tilde{A}A$. Since $(C^{-1}A)(\tilde{A}C)(C^{-1}A) = C^{-1}A$, $(\tilde{A}C)(C^{-1}A)(\tilde{A}C) = \tilde{A}C$, $[(C^{-1}A)(\tilde{A}C)]^s = (A\tilde{A})^s = A\tilde{A} = (C^{-1}A)(\tilde{A}C)$, and $[(\tilde{A}C)(C^{-1}A)]^s = (\tilde{A}A)^s = \tilde{A}A = (\tilde{A}C)(C^{-1}A)$, it follows that $\tilde{A}C$ is the Moore-Penrose inverse of $C^{-1}A$ relative to f. Similarly, $C\tilde{A}$ is the Moore-Penrose inverse of AC^{-1} relative to f. But $C^{-1}A = AC^{-1}$, and the uniqueness of the Moore-Penrose inverse implies $\tilde{A}C = C\tilde{A}$. Hence $\tilde{A}$ is also circulant. ∎

5. A METHOD OF CALCULATION OF THE MOORE-PENROSE INVERSE OF A MATRIX RELATIVE TO AN INVOLUTORY FUNCTION

Let f be an involutory function defined on the set of all matrices over a field F. If A is an $m \times n$ matrix then $f(A) = A^s$ is an $n \times m$ matrix. Let Q be an $m \times m$ invertible matrix such that $B = A^sQ$ is the column reduced form of A^s (Q can be obtained by the column operation on A^s). Let $C = AB$ and D be the row reduced form of B. The $m \times m$ matrix $E = C + Q(I_m - D^tD)$ is invertible if and only if A has a Moore-Penrose inverse relative to f. If E is invertible, then $\tilde{A} = BE^{-1}$ is the Moore-Penrose inverse of A relative to f.

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